Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Sound and Vibration 272 (2004) 361–383
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

On the dynamic plane problem of a rectangle

P. De Sabbata

Università di Urbino, Facoltà di Scienze Ambientali, Campus Scientifico-ex Sogesta, Località Crocicchie, 61029 Urbino, Italy

Received 27 November 2001; accepted 20 March 2003

Abstract

An analytical solution for the elastic plane dynamic problem of a rectangle is presented. The following boundary conditions were considered: two free edges, arbitrary normal pressure and shear displacement on the two other edges. By means of space and time Fourier transform, the solution is reduced to a set of fundamental solutions which are related to different symmetries. The solutions are expressed by means of infinite summation series involving implicit real roots. The inversion of the time Fourier transform and the numerical convergence of the solutions are discussed. As regards the applications, the solution is a tool for computations involving some kind of shock-absorber.

1. Introduction

Since the publication of Leissa's work [1] the problem of the vibrations of rectangles has been extensively investigated [2,3]. Most of the publications deal with bending vibrations because of their vital role in several applications, all of them making use of numerical methods. The effect of longitudinal displacement on the bending modes was examined by Leissa [4] to the second order. The so-called plane dynamic problem deals exclusively with longitudinal vibrations. A well-known analytical method for the solution of the plane static problem for circumference and other shapes, but not for rectangles, was found by Muskhelishvili [5]; however it is not extendable to the dynamic problem. Some formulae for the dynamic problem may be found in Radok [6], but there is no method for inserting the boundary conditions. In the literature there are no analytical solutions for the rectangle plane problem corresponding to rather general boundary conditions, even for the static case.

This paper deals with the analytical solution of the plane dynamic problem for a rectangle with two free edges and with normal pressure and tangential displacement boundary conditions on the

E-mail address: m.letizia@uniurb.it (P. De Sabbata).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2003.03.010

other edges. Boundary conditions may easily be replaced by normal displacement and tangential pressure, but not by normal and tangential pressure, which would be the most important case for practical applications. However, the solution presented can be used in order to obtain a linear combination in which the normal pressure is given and the tangential pressure is approximated by numerical methods. The generalization from two free edges to edges with constant pressure is trivial.

2. Formulation and solution of the problem

2.1. Boundary conditions

The plane stress problem (see for example, Ref. [7]), involves bodies with small thickness and constant section in the z direction. The external $z =$ constant surfaces are free and the mean value in the z direction of the external pressures acting on the other sides is orthogonal to the z axes; this means that the external force per unit length of the boundary lies in the xy plane. Then the dynamical equations of the theory of the elasticity can be reduced to a two-dimensional formulation:

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x}(x,y,\omega) + \frac{\partial \tau}{\partial y}(x,y,\omega) &= -\rho\omega^2 u_x(x,y,\omega), \\ \frac{\partial \tau}{\partial x}(x,y,\omega) + \frac{\partial \sigma_y}{\partial y}(x,y,\omega) &= -\rho\omega^2 u_y(x,y,\omega), \end{aligned} \tag{1}$$

where

$$\begin{pmatrix} \sigma_x & \tau \\ \tau & \sigma_y \end{pmatrix},$$

is the stress tensor in the xy plane. The stress tensor is connected to the displacements u_x and u_y by the equations

$$\begin{aligned} \sigma_x(x,y,\omega) &= (\lambda + 2G)\frac{\partial u_x}{\partial x}(x,y,\omega) + \lambda \frac{\partial u_y}{\partial y}(x,y,\omega), \\ \sigma_y(x,y,\omega) &= (\lambda + 2G)\frac{\partial u_y}{\partial y}(x,y,\omega) + \lambda \frac{\partial u_x}{\partial x}(x,y,\omega), \\ \tau &= G\left(\frac{\partial u_x}{\partial y}(x,y,\omega) + \frac{\partial u_y}{\partial x}(x,y,\omega)\right), \end{aligned} \tag{2}$$

where λ and G are the modified Lamé coefficients for the plane stress problem which may be expressed in terms of the three-dimensional Lamé coefficients λ^* and μ: $\lambda = 2\lambda^*\mu/(\lambda^* + 2\mu)$; $G = \mu$. In Eqs. (1), ρ is the mass density and ω is the variable of the time Fourier transform that is understood to be performed. Hereafter, Eqs. (1) and (2) are defined on a rectangular shaped domain delimited by the edges $x = \pm L/2$ and $y = \pm L_2/2$. Usually, the boundary conditions are given in terms of the stress tensor or in terms of the displacements. In the first case, σ_x and τ are given on the $x = \pm L/2$ edges while σ_y and τ are given on the $y = \pm L_2/2$ edges. In the second case, u_x and u_y are given on the four edges. There are thus eight boundary conditions.

Unfortunately, a solution with boundary conditions of this kind is not available. In order to find solutions, the eight boundary conditions must be replaced. It is supposed to have zero pressure on two edges

$$\sigma_x\left(\pm\frac{L}{2}, y, \omega\right) = 0, \tag{3a}$$

$$\tau\left(\pm\frac{L}{2}, y, \omega\right) = 0, \tag{3b}$$

while normal pressure $P_{y1}(x,\omega), P_{y2}(x,\omega)$ and tangential displacement $s_{x1}(x,\omega), s_{x2}(x,\omega)$ on the other two edges are given by

$$\begin{aligned}
&\sigma_y\left(x, \frac{L_2}{2}, \omega\right) = P_{y1}(x,\omega), \quad \sigma_y\left(x, -\frac{L_2}{2}, \omega\right) = -P_{y2}(x,\omega),\\
&u_x\left(x, \frac{L_2}{2}, \omega\right) = s_{x1}(x,\omega), \quad u_x\left(x, -\frac{L_2}{2}, \omega\right) = s_{x2}(x,\omega).
\end{aligned} \tag{4}$$

The following notation is adopted for uniformity reasons

$$\chi_1 = u_x, \quad \chi_2 = \sigma_y, \quad \chi_3 = u_y, \quad \chi_4 = \tau, \quad \chi_5 = \sigma_x.$$

2.2. Space symmetries

It is useful to introduce the x and y axes reversal operators $\hat{\mathbf{P}}_x$ and $\hat{\mathbf{P}}_y$ in order to deal with boundary conditions assigned on opposite edges. $\hat{\mathbf{P}}_x$ and $\hat{\mathbf{P}}_y$ are defined by

$$\begin{aligned}
&\hat{\mathbf{P}}_x\chi_1(x,y,\omega) = -\chi_1(-x,y,\omega), \quad \hat{\mathbf{P}}_y\chi_1(x,y,\omega) = \chi_1(x,-y,\omega),\\
&\hat{\mathbf{P}}_x\chi_2(x,y,\omega) = \chi_2(-x,y,\omega), \quad \hat{\mathbf{P}}_y\chi_2(x,y,\omega) = \chi_2(x,-y,\omega),\\
&\hat{\mathbf{P}}_x\chi_3(x,y,\omega) = \chi_3(-x,y,\omega), \quad \hat{\mathbf{P}}_y\chi_3(x,y,\omega) = -\chi_3(x,-y,\omega),\\
&\hat{\mathbf{P}}_x\chi_4(x,y,\omega) = -\chi_4(-x,y,\omega), \quad \hat{\mathbf{P}}_y\chi_4(x,y,\omega) = -\chi_4(x,-y,\omega),\\
&\hat{\mathbf{P}}_x\chi_5(x,y,\omega) = \chi_5(-x,y,\omega), \quad \hat{\mathbf{P}}_y\chi_5(x,y,\omega) = \chi_5(x,-y,\omega).
\end{aligned} \tag{5}$$

Transformations for the boundary conditions can be taken from Eqs. (4) and (5)

$$\begin{aligned}
&\hat{\mathbf{P}}_x P_{yi}(x,\omega) = P_{yi}(-x,\omega), \quad i = 1,2, \qquad \hat{\mathbf{P}}_x s_{xi}(x,\omega) = -s_{xi}(-x,\omega), \quad i = 1,2,\\
&\hat{\mathbf{P}}_y P_{y1}(x,\omega) = -P_{y2}(x,\omega), \quad \hat{\mathbf{P}}_y P_{y2}(x,\omega) = -P_{y1}(x,\omega),\\
&\hat{\mathbf{P}}_y s_{x1}(x,\omega) = s_{x2}(x,\omega), \quad \hat{\mathbf{P}}_y s_{x2}(x,\omega) = s_{x1}(x,\omega).
\end{aligned} \tag{6}$$

Symmetrized and antisymmetrized quantities $g(x,y)$ are denoted by

$$\begin{aligned}
&g^{SS} = \frac{(1+\mathbf{P}_x)(1+\mathbf{P}_y)}{4}g, \quad g^{SA} = \frac{(1+\mathbf{P}_x)(1-\mathbf{P}_y)}{4}g,\\
&g^{AS} = \frac{(1-\mathbf{P}_x)(1+\mathbf{P}_y)}{4}g, \quad g^{AA} = \frac{(1-\mathbf{P}_x)(1-\mathbf{P}_y)}{4}g
\end{aligned}$$

so that

$$\hat{\mathbf{P}}_{\mathbf{x}} g^{SB_2} = g^{SB_2}, \quad \hat{\mathbf{P}}_{\mathbf{x}} g^{AB_2} = -g^{AB_2}, \quad \hat{\mathbf{P}}_y g^{B_1 S} = g^{B_1 S}, \quad \hat{\mathbf{P}}_y g^{B_1 A} = -g^{B_1 A}$$
$$\text{with } B_j = S, A \text{ for } j = 1, 2. \tag{7}$$

Since Eqs. (1) and (2) are linear and invariant under space axis reversals the solution can be decomposed with $\chi_i = \chi_i^{SS} + \chi_i^{SA} + \chi_i^{AS} + \chi_i^{AA} = \sum_{B_1,B_2} \chi_i^{B_1 B_2}$, with $B_j = S, A$ for $j = 1, 2$ and $i = 1, 2, 3, 4, 5$.

Boundary conditions are now

$$\chi_2^{B_1 B_2}\left(x, \frac{L_2}{2}, \omega\right) = P_{y1}^{B_1 B_2}(x, \omega) \quad \chi_1^{B_1 B_2}\left(x, \frac{L_2}{2}, \omega\right) = s_{x1}^{B_1 B_2}(x, \omega), \tag{8}$$

and

$$\chi_4^{B_1 B_2}\left(\frac{L}{2}, y, \omega\right) = \chi_5^{B_1 B_2}\left(\frac{L}{2}, y, \omega\right) = 0, \tag{9}$$

while for $y = -L_2/2$ and $x = -L/2$ are identically satisfied, as a consequence of Eqs. (5)–(7). As an example, this may be checked for the $\chi_2^{B_1 B_2}$ component

$$\chi_2^{B_1 B_2}\left(x, -\frac{L_2}{2}, \omega\right) = \pm \hat{\mathbf{P}}_{\mathbf{y}} \chi_2^{B_1 B_2}\left(x, \frac{L_2}{2}, \omega\right) = \pm \hat{\mathbf{P}}_{\mathbf{y}} P_{y1}^{B_1 B_2}(x, \omega) = -P_{y2}^{B_1 B_2}(x, \omega).$$

The next step is to expand Eq. (8) in Fourier series

$$\begin{aligned} P_{y1}^{S,B_2}(x, \omega) &= \sum_{p \geqslant 0} c_{p,(2)}^{SB_2}(\omega) \cos(k_p x), \quad s_{x1}^{S,B_2}(x, \omega) = \sum_{p > 0} c_{p,(1)}^{SB_2}(\omega) \sin(k_p x), \\ P_{y1}^{A,B_2}(x, \omega) &= \sum_{p \geqslant 0} c_{p,(2)}^{AB_2}(\omega) \sin(k_p^A x), \quad s_{x1}^{A,B_2}(x, \omega) = \sum_{p \geqslant 0} c_{p,(1)}^{AB_2}(\omega) \cos(k_p^A x), \end{aligned} \tag{10}$$

where $k_p = (2\pi/L)p$ and $k_p^A = (2\pi/L)\left(p + \frac{1}{2}\right)$. It should be noted that the series involving k_p^A are modified Fourier series; however it can be demonstrated (see Appendix A) that the usual inversion formulae hold

$$c_{p,(2)}^{AB_2}(\omega) = \frac{2}{L} \int_{-L/2}^{L/2} P_{y1}^{AB_2}(x, \omega) \sin(k_p^A x)\, \mathrm{d}x \quad \text{and} \quad c_{p,(1)}^{AB_2}(\omega) = \frac{2}{L} \int_{-L/2}^{L/2} s_{x1}^{AB_2}(x, \omega) \cos(k_p^A x)\, \mathrm{d}x. \tag{11}$$

(The reasons for taking k_p^A instead of k_p in two equations are connected to further developments and cannot be elucidated at this stage.) Then the solutions are given by

$$\chi_i = \sum_{p \in \mathbf{N}} \sum_{B_1 B_2} c_{p,(1)}^{B_1 B_2}(\omega) \chi_{ip(1)}^{B_1 B_2}(x, y, \omega) + c_{p,(2)}^{B_1 B_2}(\omega) \chi_{ip(2)}^{B_1 B_2}(x, y, \omega), \tag{12}$$

in which the functions $\chi_{ip(j)}^{B_1B_2}$, $j = 1, 2$; $B_1, B_2 = A, S$ satisfy both the boundary conditions

$$\begin{aligned}
&\chi_{1p(1)}^{SB_2}\left(x, \frac{L_2}{2}, \omega\right) = \sin(k_p x) \quad \chi_{2p(1)}^{SB_2}\left(x, \frac{L_2}{2}, \omega\right) = 0,\\
&\chi_{1p(2)}^{SB_2}\left(x, \frac{L_2}{2}, \omega\right) = 0 \quad \chi_{2p(2)}^{SB_2}\left(x, \frac{L_2}{2}, \omega\right) = \cos(k_p x),\\
&\chi_{1p(1)}^{AB_2}\left(x, \frac{L_2}{2}, \omega\right) = \cos(k_p^A x) \quad \chi_{2p(1)}^{AB_2}\left(x, \frac{L_2}{2}, \omega\right) = 0,\\
&\chi_{1p(2)}^{AB_2}\left(x, \frac{L_2}{2}, \omega\right) = 0 \quad \chi_{2p(2)}^{AB_2}\left(x, \frac{L_2}{2}, \omega\right) = \sin(k_p^A x)
\end{aligned} \tag{13}$$

and conditions (9). Therefore the functions $\chi_{ip(j)}^{B_1B_2}$ are the fundamental solutions to which all the other solutions may be constructed.

2.3. The fundamental solutions

This sub-paragraph deals with the fundamental solutions $\chi_{ip(j)}^{B_1B_2}$ defined from conditions (9) and (13). Eqs. (1) and (2) can be considered as differential equations with respect to the y variable. Since there are no $\partial_y \chi_5$ terms, χ_5 must be considered as a dependent variable:

$$\sigma_x = \chi_5 = \frac{\lambda}{(\lambda + 2G)}\chi_2 + \frac{4G(G+\lambda)}{(\lambda+2G)}\partial_x \chi_1. \tag{14}$$

Taking into account Eq. (14), Eqs. (1) and (2) can be rewritten as

$$\partial_y \chi_i = \mathbf{A}_{ij}\chi_j, \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & -\partial_x & \dfrac{1}{G} \\ 0 & 0 & -\rho\omega^2 & -\partial_x \\ \dfrac{-\lambda}{(\lambda+2G)}\partial_x & \dfrac{1}{(\lambda+2G)} & 0 & 0 \\ \left(\dfrac{-4G(\lambda+G)}{(\lambda+2G)}\partial_x^2 - \omega^2\rho\right) & \dfrac{-\lambda}{(\lambda+2G)}\partial_x & 0 & 0 \end{pmatrix}, \tag{15}$$

with $1 \leqslant i,j \leqslant 4$.

Following Eqs. (5), a solution with symmetry χ_i^{SS} can be found by putting

$$\begin{aligned}
&\chi_1^{SS}(x,y,\omega) = c_1(\omega,z)\sin(\xi x)\cosh(zy), \quad \chi_2^{SS}(x,y,\omega) = c_2(\omega,z)\cos(\xi x)\cosh(zy),\\
&\chi_3^{SS}(x,y,\omega) = c_3(\omega,z)\cos(\xi x)\sinh(zy), \quad \chi_4^{SS}(x,y,\omega) = c_4(\omega,z)\sin(\xi x)\sinh(zy).
\end{aligned} \tag{16}$$

The substitution of Eqs. (16) into Eq. (15) leads to a system of algebraic equations in the c_i variables which admits non-trivial solutions only for the two eigenvalues

$$\xi_1 = \xi_1(z,\omega) = \sqrt{z^2 + \frac{\omega^2\rho}{(\lambda+2G)}}, \quad \xi_2 = \xi_2(z,\omega) = \sqrt{z^2 + \frac{\omega^2\rho}{G}}.$$

The two corresponding solutions χ_i bear the same y dependence, and therefore it is easy to obtain a linear combination for which Eq. (3b) holds; this combination is given by

$$\chi_1^{SS}(x,y,\omega,z) = \xi_1\xi_2\left[-\left(z^2+\frac{\omega^2\rho}{2G}\right)\sin\left(\xi_2\frac{L}{2}\right)\sin(\xi_1 x) + z^2\sin\left(\xi_1\frac{L}{2}\right)\sin(\xi_2 x)\right]\cosh(zy),$$

$$\chi_2^{SS}(x,y,\omega,z) = 2G\xi_2\left[\left(z^2-\frac{\lambda\omega^2\rho}{2G(\lambda+2G)}\right)\left(z^2+\frac{\omega^2\rho}{2G}\right)\sin\left(\xi_2\frac{L}{2}\right)\cos(\xi_1 x) - \xi_1\xi_2 z^2\sin\left(\xi_1\frac{L}{2}\right)\cos(\xi_2 x)\right]\cosh(zy),$$

$$\chi_3^{SS}(x,y,\omega,z) = z\xi_2\left[\left(z^2+\frac{\omega^2\rho}{2G}\right)\sin\left(\xi_2\frac{L}{2}\right)\cos(\xi_1 x) - \xi_1\xi_2\sin\left(\xi_1\frac{L}{2}\right)\cos(\xi_2 x)\right]\sinh(zy),$$

$$\chi_4^{SS}(x,y,\omega,z) = 2Gz\xi_1\xi_2\left(z^2+\frac{\omega^2\rho}{2G}\right)\left(-\sin\left(\xi_2\frac{L}{2}\right)\sin(\xi_1 x) + \sin\left(\xi_1\frac{L}{2}\right)\sin(\xi_2 x)\right)\sinh(zy). \tag{17a}$$

Eq. (3a), and taking into account Eq. (14), leads to the equation $\phi^S(z,\omega)=0$, where

$$\phi^S(\omega,z) = z^2\xi_1\sin\left(\xi_1\frac{L}{2}\right)\cos\left(\xi_2\frac{L}{2}\right) - \frac{\left(z^2+\frac{\omega^2\rho}{2G}\right)^2\sin\left(\xi_2\frac{L}{2}\right)}{\xi_2}\cos\left(\xi_1\frac{L}{2}\right).$$

There are infinite complex roots z_α^S, $\alpha = \pm1, \pm2, \pm3\ldots$ for this equation. Therefore there are an infinity of solutions χ_i^{SS} depending on a complex parameter z_α^S: $\chi_i^{SS} = \chi_i^{SS}(x,y,\omega,z_\alpha^S)$. Hereafter the functions are referred to complex variables so that, for instance: $\sin(\mathrm{i}zy) = \mathrm{i}\sinh(zy)$, $\sqrt{z^2+\omega^2\rho/2G} = \pm\mathrm{i}\sqrt{-z^2-\omega^2\rho/2G}$ and so on. Likewise solutions bearing the other symmetries can be found:

$$\chi_i^{SA}(x,y,\omega,z) = \frac{\mathrm{i}}{z}\partial_y\chi_i^{SS}(x,y,\omega,z),$$

$$\chi_1^{AS}(x,y,\omega,z) = \xi_1\xi_2\left[-\left(z^2+\frac{\omega^2\rho}{2G}\right)\cos\left(\xi_2\frac{L}{2}\right)\cos(\xi_1 x) + z^2\cos\left(\xi_1\frac{L}{2}\right)\cos(\xi_2 x)\right]\cosh(zy),$$

$$\chi_2^{AS}(x,y,\omega,z) = 2G\xi_2\left[-\left(z^2-\frac{\lambda\omega^2\rho}{2G(\lambda+2G)}\right)\left(z^2+\frac{\omega^2\rho}{2G}\right)\cos\left(\xi_2\frac{L}{2}\right)\sin(\xi_1 x) + \xi_1\xi_2 z^2\cos\left(\xi_1\frac{L}{2}\right)\sin(\xi_2 x)\right]\cosh(zy),$$

$$\chi_3^{AS}(x,y,\omega,z) = z\xi_2\left[-\left(z^2+\frac{\omega^2\rho}{2G}\right)\cos\left(\xi_2\frac{L}{2}\right)\sin(\xi_1 x) + \xi_1\xi_2\cos\left(\xi_1\frac{L}{2}\right)\sin(\xi_2 x)\right]\sinh(zy),$$

$$\chi_4^{AS}(x,y,\omega,z) = 2Gz\xi_1\xi_2\left(z^2+\frac{\omega^2\rho}{2G}\right)\left(-\cos\left(\xi_2\frac{L}{2}\right)\cos(\xi_1 x) + \cos\left(\xi_1\frac{L}{2}\right)\cos(\xi_2 x)\right)\sinh(zy),$$

$$\chi_i^{AA}(x,y,\omega,z) = \frac{\mathrm{i}}{z}\partial_y\chi_i^{AS}(x,y,\omega,z). \tag{17b}$$

The solutions compatible with condition (3a) are given by $\chi_i^{SA} = \chi_i^{SA}(x, y, \omega, z_\alpha^S)$; $\chi_i^{AB_2} = \chi_i^{AB_2}(x, y, \omega, z_\alpha^A)$; here the z_α^A are the roots of the equation

$$\phi^A(\omega, z) = z^2 \xi_1 \cos\left(\xi_1 \frac{L}{2}\right) \sin\left(\xi_2 \frac{L}{2}\right) - \frac{\left(z^2 + \frac{\omega^2\rho}{2G}\right)^2 \cos\left(\xi_2 \frac{L}{2}\right)}{\xi_2} \sin\left(\xi_1 \frac{L}{2}\right) = 0.$$

Now boundary conditions (13) must be taken into account. The first to be considered is the solution $\chi_{ip(1)}^{SS}(x, y, \omega)$. The functions

$$f_{ip}^{(1)}(z) = \frac{z}{(\xi_1^2 - k_p^2)} \frac{1}{\xi_2^2} \frac{1}{\phi^S(z, \omega)} \frac{\chi_i^{SS}(x, y, \omega, z)}{\cosh\left(z \frac{L_2}{2}\right)}, \quad i = 1, 2, 3, 4, \tag{18}$$

are invariant under the substitutions $\xi_1 \mapsto -\xi_1$ and $\xi_2 \mapsto -\xi_2$, therefore they are single-valued functions in the complex variable z. The integrals $\int_{C_R} f_{ip}^{(1)}(z)\,\mathrm{d}z$ have a vanishing limit if the integration path C_R tends to infinity avoiding the neighbourhood of the poles of the function. In fact, if $|x| < L/2$ and $|y| < L_2/2$ the goniometrical and hyperbolical functions which takes place in the numerator through the χ_i^{SS} approach infinity faster than those that take place in the denominator through the factor ϕ^S. The limit to be performed is of the kind of the limits of the ratios

$$\frac{\sin\left(\xi_2 \frac{L}{2}\right)\sin(\xi_1 x)}{\sin\left(\xi_1 \frac{L}{2}\right)\cos\left(\xi_2 \frac{L}{2}\right)}, \quad \frac{\cosh(zy)}{\cosh\left(z \frac{L_2}{2}\right)},$$

and so on. It should be noted that at least one of these two ratios has an exponential convergence for $z \to \infty$ in the complex plane. In the other case, $x = \pm L/2$ or $y = \pm L_2/2$ the functions $f_{ip}^{(1)}(z)$ are convergent at least as $1/z^2$, even on the real and imaginary axis. As a consequence of the residue theory it is apparent that

$$\sum_{z_\alpha^S} res\{f_{ip}^{(1)}(z)\} = - \sum_{\substack{all\ the\\ other\ poles}} res\{f_{ip}^{(1)}(z)\}. \tag{19}$$

The left side of this equation can be written as

$$\sum_\alpha A_{p\alpha}^{(1)} \chi_i^{SS}(x, y, \omega, z_\alpha^S), \tag{20}$$

with

$$A_{p\alpha}^{(1)} = \frac{z_\alpha^S}{[(\xi_{1\alpha}^S)^2 - k_p^2]} \frac{1}{(\xi_{2\alpha}^S)^2} \frac{1}{\partial_z \phi^S(\omega, z_\alpha^S)} \frac{1}{\cosh\left(z_\alpha^S \frac{L_2}{2}\right)},$$

and

$$\xi_{1\alpha}^S = \xi_{1\alpha}^S(\omega) = \xi_1(z_\alpha^S, \omega) = \sqrt{(z_\alpha^S)^2 + \frac{\omega^2\rho}{(\lambda + 2G)}}, \quad \xi_{2\alpha}^S = \xi_{2\alpha}^S(\omega) = \xi_2(z_\alpha^S, \omega) = \sqrt{(z_\alpha^S)^2 + \frac{\omega^2\rho}{G}}.$$

Series (20) is a solution for which both conditions (3) hold, since z_α^S is a root of the function $\phi^S(\omega, z)$. This solution is denoted by $\chi_{ip}^{SS(1)}(x, y, \omega)$, and can be taken from the right-hand side of Eq. (19), obtaining, if $i = 1$ and $i = 2$,

$$
\begin{aligned}
\chi_{1p}^{SS(1)} = & \frac{(-1)^{p+1} k_p}{k_p^2 - \frac{\omega^2\rho}{\lambda+2G} + \frac{\omega^2\rho}{2G}} \frac{\cosh(z_{1p} y)\sin(k_p x)}{\cosh\left(z_{1p}\frac{L_2}{2}\right)} \\
& - \sum_{k=-\infty}^{+\infty} \frac{(-1)^k 2}{L_2} \frac{s_k^S}{(\xi_{1k}^S)^2 - k_p^2} \frac{1}{(\xi_{2k}^S)^2} \frac{1}{\phi^S(\omega, \mathrm{i}s_k^S)} \chi_1^{SS}(x, y, \omega, \mathrm{i}s_k^S),
\end{aligned}
$$

$$
\begin{aligned}
\chi_{2p}^{SS(1)} = & \frac{(-1)^p 2G\left(k_p^2 - \frac{\omega^2\rho}{2G}\right)}{k_p^2 - \frac{\omega^2\rho}{\lambda+2G} + \frac{\omega^2\rho}{2G}} \frac{\cosh(z_{1p} y)\cos(k_p x)}{\cosh\left(z_{1p}\frac{L_2}{2}\right)} \\
& - \sum_{k=-\infty}^{+\infty} \frac{(-1)^k 2}{L_2} \frac{s_k^S}{(\xi_{1k}^S)^2 - k_p^2} \frac{1}{(\xi_{2k}^S)^2} \frac{1}{\phi^S(\omega, \mathrm{i}s_k^S)} \chi_2^{SS}(x, y, \omega, \mathrm{i}s_k^S). \qquad (21)
\end{aligned}
$$

Here the following notations were adopted

$$
s_k^S = \frac{2\pi}{L_2}\left(k + \frac{1}{2}\right), \quad \xi_{1k}^S = \xi_1(\mathrm{i}s_k^S, \omega) = \sqrt{\frac{\omega^2\rho}{\lambda + 2G} - (s_k^S)^2},
$$

$$
\xi_{2k}^S = \xi_2(\mathrm{i}s_k^S, \omega) = \sqrt{\frac{\omega^2\rho}{G} - (s_k^S)^2}, \quad z_{1p} = \sqrt{k_p^2 - \frac{\omega^2\rho}{\lambda + 2G}}.
$$

It should be noted that the terms involving series in Eqs. (21) vanish if $y = L_2/2$ owing to the presence of the null factors $\chi_i^{SS}(x, L_2/2, \omega, \mathrm{i}s_k^S) = 0,\ i = 1, 2$. Another solution can be derived by taking, instead of Eq. (18), the functions

$$
f_{ip}^{(2)}(z) = \frac{z}{(\xi_2^2 - k_p^2)} \frac{1}{\xi_2^2} \frac{1}{\phi^S(z, \omega)} \frac{\chi_i^{SS}(x, y, \omega, z)}{\cosh\left(z\frac{L_2}{2}\right)}, \quad i = 1, 2, 3, 4.
$$

This leads to the solution $\chi_{ip}^{SS(2)} = \sum_\alpha A_{p\alpha}^{(2)} \chi_i^{SS}(x, y, \omega, z_\alpha^S)$, that is for $i = 1$ e 2 given by

$$
\chi_{1p}^{SS(2)} = \frac{(-1)^{p+1}}{k_p} \frac{\cosh(z_{2p} y)\sin(k_p x)}{\cosh\left(z_{2p}\frac{L_2}{2}\right)} - \sum_{k=-\infty}^{+\infty} \frac{(-1)^k 2}{L_2} \frac{s_k^S}{[(\xi_{2k}^S)^2 - k_p^2]} \frac{1}{(\xi_{2k}^S)^2} \frac{1}{\phi^S(\omega, \mathrm{i}s_k^S)} \chi_1^{SS}(x, y, \omega, \mathrm{i}s_k^S),
$$

$$
\chi_{2p}^{SS(2)} = (-1)^p 2G \frac{\cosh(z_{2p} y)\cos(k_p x)}{\cosh\left(z_{2p}\frac{L_2}{2}\right)} - \sum_{k=-\infty}^{+\infty} \frac{(-1)^k 2}{L_2} \frac{s_k^S}{[(\xi_{2k}^S)^2 - k_p^2]} \frac{1}{(\xi_{2k}^S)^2} \frac{1}{\phi^S(\omega, \mathrm{i}s_k^S)} \chi_2^{SS}(x, y, \omega, \mathrm{i}s_k^S). \qquad (22)
$$

Here the notation $z_{2p} = \sqrt{k_p^2 - \omega^2\rho/G}$ was introduced. If $y = L_2/2$, since the series are vanishing, both solutions (21) and (22) bear the same x-dependence for the $i = 1$ and $i = 2$ components: the factor $\sin(k_p x)$ for $\chi_{1p}^{SS(j)}(x, L_2/2, \omega),\ j = 1, 2$ and the factor $\cos(k_p x)$ for $\chi_{2p}^{SS(j)}(x, L_2/2, \omega)$,

$j = 1, 2$. Therefore, it is easy to find a linear combination which fulfils the boundary conditions (13), that are

$$\chi^{SS}_{1p(1)}\left(x, \frac{L_2}{2}, \omega\right) = \sin(k_p x), \quad \chi^{SS}_{2p(1)}\left(x, \frac{L_2}{2}, \omega\right) = 0, \tag{23a}$$

or

$$\chi^{SS}_{1p(2)}\left(x, \frac{L_2}{2}, \omega\right) = 0 \quad \chi^{SS}_{2p(2)}\left(x, \frac{L_2}{2}, \omega\right) = \cos(k_p x). \tag{23b}$$

Now consider the first of these two boundary conditions: the $i = 1$ component of the related solution is

$$\begin{aligned}\chi^{SS}_{1p(1)} = &\frac{2G}{\omega^2\rho}\left(k_p^2 \frac{\cosh(z_{1p}y)}{\cosh\left(z_{1p}\frac{L_2}{2}\right)} - \left(k_p^2 - \frac{\omega^2\rho}{2G}\right)\frac{\cosh(z_{2p}y)}{\cosh\left(z_{2p}\frac{L_2}{2}\right)}\right)\sin(k_p x) \\ &+ \sum_{k\in\mathbf{N}} (-1)^{p+k} 8Gk_p s_k^S E_{kp}(\omega)\frac{\chi_1^{SS}(x, y, \omega, \mathrm{i}s_k^S)}{L_2(\xi^S_{2k})^2\omega^2\rho\phi^S(\omega, \mathrm{i}s_k^S)},\end{aligned} \tag{24}$$

where

$$E_{kp}(\omega) = \frac{\left\{k_p^2 - \frac{\omega^2\rho}{(\lambda+2G)} + \frac{\omega^2\rho}{2G}\right\}}{\{[\xi^S_{1k}(\omega)]^2 - k_p^2\}} - \frac{\left\{k_p^2 - \frac{\omega^2\rho}{2G}\right\}}{\{[\xi^S_{2k}(\omega)]^2 - k_p^2\}}.$$

In the limit $k \to \infty$, the series in Eq. (24) behaves asymptotically as

$$\sum_{k\in\mathbf{N}}(\ldots) \approx \sum_{k\in\mathbf{N}} (-1)^{p+k+1}\frac{8}{L_2 s_k^S}\Lambda k_p \gamma_k(x)\,\mathrm{sign}(x)\left(\frac{L}{2} - |x|\right)\cos(s_k^S y),$$

where

$$\Lambda = \frac{(\lambda + G)}{(\lambda + 2G)} \quad \text{and} \quad \gamma_k(x) = \mathrm{e}^{s_k^S(|x| - L/2)}.$$

This quantity may be subtracted in the series when adding its sum which may be performed analytically. This gives, if $|y| < L_2/2$ and $|x| < L/2$,

$$\begin{aligned}\chi^{SS}_{1p(1)} = &\frac{2G}{\omega^2\rho}\left(k_p^2 r_{1p}\cosh(z_{1p}y) - \left(k_p^2 - \frac{\omega^2\rho}{2G}\right)r_{2p}\cosh(z_{2p}y)\right)\sin(k_p x) \\ &+ (-1)^{p+1}\Lambda 4k_p\,\mathrm{sign}(x)\left(\frac{L}{2} - |x|\right)\frac{1}{\pi}\mathrm{arctg}\left(\tau^*\cos\left(\frac{\pi}{L_2}y\right)\right) \\ &+ \sum_{k\in\mathbf{N}}(-1)^{p+k}\frac{8k_p}{L_2}\left[Gs_k^S E_{kp}(\omega)\frac{\chi_1^{SS}(x, y, \omega, \mathrm{i}s_k^S)}{(\xi^S_{2k})^2\omega^2\rho\phi^S(\omega, \mathrm{i}s_k^S)} + \Lambda\gamma_k(x)\,\mathrm{sign}(x)\frac{1}{s_k^S}\left(\frac{L}{2} - |x|\right)\cos(s_k^S y)\right],\end{aligned} \tag{25a}$$

where

$$\tau = \mathrm{e}^{\pi/L_2(|x|-L/2)}, \quad \tau^* = \frac{2\tau}{(1-\tau^2)}, \quad r_{1p} = \frac{1}{\cosh\left(z_{1p}\frac{L_2}{2}\right)} \quad \text{and} \quad r_{2p} = \frac{1}{\cosh\left(z_{2p}\frac{L_2}{2}\right)}.$$

The other components of this solution are

$$\begin{aligned}\chi^{SS}_{2p(1)} =& \frac{-4G^2k_p}{\omega^2\rho}k^2_{p2G}(r_{1p}\cosh(z_{1p}y) - r_{2p}\cosh(z_{2p}y))\cos(k_p x)\\ &+ (-1)^{p+1}8G\Lambda k_p\left[\frac{1}{\pi}\mathrm{arctg}\left(\tau^*\cos\left(\frac{\pi}{L_2}y\right)\right) + \frac{1}{L_2}\frac{\left(|x|-\frac{L}{2}\right)2\tau(1+\tau^2)}{1+\tau^4+2\tau^2\cos\left(\frac{2\pi}{L_2}y\right)}\cos\left(\frac{\pi}{L_2}y\right)\right]\\ &+ \sum_{k\in\mathbf{N}}(-1)^{p+k}\frac{8Gk_p}{L_2}\left[s^S_k E_{kp}(\omega)\frac{\chi^{SS}_2(x,y,\omega,\mathrm{i}s^S_k)}{(\xi^S_{2k})^2\omega^2\rho\phi^S(\omega,\mathrm{i}s^S_k)} + 2\Lambda\gamma_k(x)\left(\frac{1}{s^S_k} - \frac{L}{2} + |x|\right)\cos(s^S_k y)\right],\end{aligned} \tag{25b}$$

$$\begin{aligned}\chi^{SS}_{3p(1)} =& -\frac{2Gk_p}{\omega^2\rho}\left(z_{1p}r_{1p}\sinh(z_{1p}y) - \frac{k^2_{p2G}}{k^2_{pG}}z_{2p}r_{2p}\sinh(z_{2p}y)\right)\cos(k_p x)\\ &+ (-1)^p 2\Lambda k_p\left(\frac{L}{2}-|x|\right)\frac{1}{\pi}\log\left(\frac{1+\tau^2+2\tau\sin\left(\frac{\pi}{L_2}y\right)}{1+\tau^2-2\tau\sin\left(\frac{\pi}{L_2}y\right)}\right)\\ &+ \sum_{k\in\mathbf{N}}(-1)^{p+k}\frac{8k_p}{L_2}\left[Gs^S_k E_{kp}(\omega)\frac{\chi^{SS}_3(x,y,\omega,\mathrm{i}s^S_k)}{(\xi^S_{2k})^2\omega^2\rho\phi^S(\omega,\mathrm{i}s^S_k)} - \Lambda\gamma_k(x)\frac{\frac{L}{2}-|x|}{s^S_k}\sin(s^S_k y)\right],\end{aligned} \tag{25c}$$

$$\begin{aligned}\chi^{SS}_{4p(1)} =& \frac{4G^2}{\omega^2\rho}\left(k^2_p z_{1p}r_{1p}\sinh(z_{1p}y) - \frac{(k^2_{p2G})^2}{k^2_{pG}}z_{2p}r_{2p}\sinh(z_{2p}y)\right)\sin(k_p x)\\ &+ \frac{(-1)^p}{L_2}8Gk_p\Lambda\,\mathrm{sign}(x)\left(\frac{L}{2}-|x|\right)\frac{2\tau(1-\tau^2)\sin\left(\frac{\pi}{L_2}y\right)}{1+\tau^4+2\tau^2\cos\left(\frac{2\pi}{L_2}y\right)}\\ &+ \sum_{k\in\mathbf{N}}(-1)^{p+k}\frac{8Gk_p}{L_2}\left[s^S_k E_{kp}(\omega)\frac{x^{SS}_4(x,y,\omega,\mathrm{i}s^S_k)}{(\xi^S_{2k})^2\omega^2\rho\phi^S(\omega,\mathrm{i}s^S_k)} - 2\Lambda\gamma_k(x)\mathrm{sign}(x)\left(\frac{L}{2}-|x|\right)\sin(s^S_k y)\right],\end{aligned} \tag{25d}$$

where $k^2_{pG} = k^2_p - \omega^2\rho/G$ and $k^2_{p2G} = k^2_p - \omega^2\rho/2G$.

Here all the series are uniformly convergent, at least as $1/(s^S_k)^2 \approx 1/k^2$; then it is apparent that in Eqs. (25a) and (25b) $\lim_{y\to\pm\frac{L_2}{2}}\sum(\ldots) = \sum(\ldots)|_{y=\frac{L_2}{2}} = 0$, due to the disappearance of the factors χ^{SS}_1 and χ^{SS}_2. Then it is easy to check that conditions (23a) are fulfilled.

2.4. Inversion of the Fourier transform

Eqs. (25) are the formulation which may be better approximated by truncation, but this is not convenient for inverting Fourier transform from the ω variable to the time t. Another formulation can be found by going back to Eq. (24). The series therein is denoted by $\sum_{k\in\mathbf{N}} g_k(\omega^2\rho)$. In the right-hand side, every function g_k is to be expanded in a series summation on its poles (see for example, Ref. [8]):

$$g_k(\omega^2\rho) = \frac{c_{kA}}{(\omega^2\rho - \omega_{kA}^2\rho)} + \frac{c_{kB}}{(\omega^2\rho - \omega_{kB}^2\rho)} + \sum_{\alpha\in\mathbf{N}} \frac{c_{k\alpha}}{(\omega^2\rho - (\omega_{k\alpha}^{SS})^2\rho)}. \tag{26}$$

Here ω_{kA}^2 and ω_{kB}^2 are the solutions of $(\xi_{1k}^S)^2 = k_p^2$ and $(\xi_{2k}^S)^2 = k_p^2$; the $\omega_{k\alpha}^{SS}$ are the roots of the equation

$$\phi^S(\omega, \mathrm{i}s_k^S) = 0, \tag{27}$$

and the $c_{kj};\ j = \alpha, A, B$ are the residues. It may be proved (see Appendix B) that all the solutions of Eq. (27) are real, so that they may be easily found by numerical methods, going step by step on the real axes. These solutions represent all the singularities of the $\chi_{3p(1)}^{SS}$ and $\chi_{4p(1)}^{SS}$ in Eqs. (25c) and (25d), that is all the "resonance frequencies" since the singularities r_{jp}, for $j = 1, 2$ which occur for $z_{jp}L_2/2 = \mathrm{i}\pi(k + 1/2),\ j = 1, 2$ and $k = 1, 2, \ldots,$ are cancelled by the singularities of the term $E_{kp}(\omega)$. After the substitution of Eq. (26) into Eq. (24) has been performed, the series

$$\sum_{k\in\mathbf{N}} \frac{c_{kA}}{(\omega^2\rho - \omega_{kA}^2\rho)} + \frac{c_{kB}}{(\omega^2\rho - \omega_{kB}^2\rho)},$$

may be summed analytically, its sum being just the opposite of the first term of Eq. (24). This leads to the formula

$$\chi_{1p(1)}^{SS} = \begin{cases} \sin(k_p x), & \text{if } y = \pm\dfrac{L_2}{2} \\ \sum_{\substack{k\in\mathbf{N}\\ \alpha\in\mathbf{N},}} \dfrac{(-1)^{k+p} 8 G k_p s_k^S}{L_2(\omega_{k\alpha}^{SS})^2 \rho (\xi_{2k\alpha}^{SS})^2} E_{kp}(\omega_{k\alpha}^{SS}) \dfrac{\chi_1^{SS}(x, y, \omega_{k\alpha}^{SS}, \mathrm{i}s_k^S)}{\phi_\omega^{S'}(\omega_{k\alpha}^{SS}, \mathrm{i}s_k^S)} \dfrac{1}{[\omega^2\rho - (\omega_{k\alpha}^{SS})^2\rho]} & \text{if } |y| < \dfrac{L_2}{2}, \end{cases} \tag{28}$$

where

$$\xi_{1k\alpha}^{SS} = \xi_1(\mathrm{i}s_k^S, \omega_{k\alpha}^{SS}) = \sqrt{\frac{(\omega_{k\alpha}^{SS})^2\rho}{\lambda + 2G} - (s_k^S)^2} \quad \text{and} \quad \xi_{2k\alpha}^{SS} = \xi_2(\mathrm{i}s_k^S, \omega_{k\alpha}^{SS}) = \sqrt{\frac{(\omega_{k\alpha}^{SS})^2\rho}{G} - (s_k^S)^2};$$

$$\phi_\omega^{S'}(\mathrm{i}s_k^S, \omega) = \frac{\partial}{\partial\zeta}\phi(\mathrm{i}s_k^S, \omega), \quad \zeta = \omega^2\rho.$$

The series in Eq. (28) is not uniformly convergent. This is a consequence of the fact that, taking into account the previous observation on Eq. (25), it can be found that

$$\sin(k_p x) = \chi_{1p(1)}^{SS}\left(x, \frac{L_2}{2}, \omega\right) = \lim_{y\to\frac{L_2}{2}} \chi_{1p(1)}^{SS}(x, y, \omega) = \lim_{y\to\frac{L_2}{2}} \sum_{\substack{k\in\mathbf{N}\\ \alpha\in\mathbf{N}}} (\ldots) \neq \sum_{\substack{k\in\mathbf{N}\\ \alpha\in\mathbf{N}}} \lim_{y\to\frac{L_2}{2}} (\ldots) = 0.$$

All the solutions in the same formulation of Eq. (28) can be found likewise:

$$\chi_{ip(j)}^{B_1B_2}(x,y,\omega)=\sum_{k=-\infty}^{+\infty}\sum_{\alpha}\frac{e_{ip(j)k\alpha}^{B_1B_2}(x,y)}{(\omega^2\rho-(\omega_{k\alpha}^{B_1B_2})^2\rho)},\qquad \begin{matrix} i=1,2,3,4, & p\in\mathbf{N} \\ j=1,2, & B_1,B_2=S,A\end{matrix}\qquad \text{if } |y|<\frac{L_2}{2}, \tag{29}$$

(if $y=\pm L_2/2$, Eqs. (13) must be used) where

$$e_{ip(1)k\alpha}^{B_1B_2}(x,y)=\frac{(-1)^{p+k}4Gk_p^{B_1}s_k^{B_2}}{L_2(\xi_{2k\alpha}^{B_1B_2})^2(\omega_{k\alpha}^{B_1B_2})^2\rho}\left(\frac{(k_p^{B_1})^2+\frac{\lambda(\omega_{k\alpha}^{B_1B_2})^2\rho}{2G(\lambda+2G)}}{[(\xi_{1k\alpha}^{B_1B_2})^2-(k_p^{B_1})^2]}-\frac{(k_p^{B_1})^2-\frac{(\omega_{k\alpha}^{B_1B_2})^2\rho}{2G}}{[(\xi_{2k\alpha}^{B_1B_2})^2-(k_p^{B_1})^2]}\right)\frac{\chi_i^{B_1B_2}(x,y,\omega_{k\alpha}^{B_1B_2},\mathrm{i}s_k^{B_2})}{\phi_{\omega}^{B_1'}(\omega_{k\alpha}^{B_1B_2},\mathrm{i}s_k^{B_2})},$$

$$e_{ip(2)k\alpha}^{B_1B_2}(x,y)=\frac{(-1)^{p+k}\varepsilon^{B_1}2s_k^{B_2}}{L_2(\xi_{2k\alpha}^{B_1B_2})^2(\omega_{k\alpha}^{B_1B_2})^2\rho}\left(\frac{(k_p^{B_1})^2+\frac{\lambda(\omega_{k\alpha}^{B_1B_2})^2\rho}{2G(\lambda+2G)}}{[(\xi_{1k\alpha}^{B_1B_2})^2-(k_p^{B_1})^2]}-\frac{(k_p^{B_1})^2}{[(\xi_{2k\alpha}^{B_1B_2})^2-(k_p^{B_1})^2]}\right)\frac{\chi_i^{B_1B_2}(x,y,\omega_{k\alpha}^{B_1B_2},\mathrm{i}s_k^{B_2})}{\phi_{\omega}^{B_1'}(\omega_{k\alpha}^{B_1B_2},\mathrm{i}s_k^{B_2})},$$

$$k_p^S=k_p=\frac{2\pi}{L}p,\quad k_p^A=\frac{2\pi}{L}\left(p+\frac{1}{2}\right),\quad s_k^S=\frac{2\pi}{L_2}\left(k+\frac{1}{2}\right),\quad s_k^A=\frac{2\pi}{L_2}k$$

$\omega_{k\alpha}^{B_1B_2}$ are the roots of $\phi^{B_1}(\omega,\mathrm{i}s_k^{B_2})=0$, $\phi_{\omega}^{B_1'}(\omega,z)=(\partial/\partial\zeta)\phi^{B_1}(\omega,z)$, $\zeta=\omega^2\rho$,

$$\xi_{1k\alpha}^{B_1B_2}=\sqrt{\frac{(\omega_{k\alpha}^{B_1B_2})^2\rho}{(\lambda+2G)}-(s_k^{B_2})^2},\quad \xi_{2k\alpha}^{B_1B_2}=\sqrt{\frac{(\omega_{k\alpha}^{B_1B_2})^2\rho}{G}-(s_k^{B_2})^2},$$

ε^{B_1} is defined by $\varepsilon^S=1$ and $\varepsilon^A=-1$.

The coefficients $e_{ip(j)0\alpha}^{B_1A}$, for $B_2=A$ and $k=0$, involving the term $s_0^A=0$, are of the kind 0/0, due to the coincidence of one of the roots $\omega_{0\alpha}^{B_1A}$ of $\phi^{B_1}(\omega_{0\alpha}^{B_1A},\mathrm{i}s_0^A)=0$ with that of the term $[\xi_{20\alpha}^{B_1A}(\omega)]^2-(k_p^{B_1})^2=0$, that is $\omega^2\rho/G=(k_p^{B_1})^2$. The ratio 0/0 therein must be intended as a limit for $s\to s_0^A=0$ and $\omega^2\rho/G\to(k_p^{B_1})^2$. It can be confirmed that the equivalence between Eq. (29) and the formulation of the solutions of the kind (25) is preserved for all the symmetry indices B_1 and B_2 only if the two limits, in the variables s and ω, are performed according to the formula

$$e_{ip(1)0\alpha}^{B_1A}(x,y)=\lim_{s\to 0}\frac{(-1)^p4Gk_p^{B_1}s}{L_2\xi_2^2(\omega(s))^2\rho}\left(\frac{(k_p^{B_1})^2+\frac{\lambda(\omega(s))^2\rho}{2G(\lambda+2G)}}{[\xi_1^2-(k_p^{B_1})^2]}-\frac{(k_p^{B_1})^2-\frac{(\omega(s))^2\rho}{2G}}{[\xi_2^2-(k_p^{B_1})^2]}\right)\frac{\chi_i^{B_1A}(x,y,\omega(s),\mathrm{i}s)}{\phi_{\omega}^{B_1}(\omega(s),\mathrm{i}s(s))},$$

in which

$$\xi_1=\sqrt{\frac{[\omega(s)]^2}{\lambda+2G}-s^2},\quad \xi_2=\sqrt{\frac{[\omega(s)]^2}{G}-s^2},$$

and $\omega(s)$ is the function defined by $\phi^{B_1}(\omega(s),\mathrm{i}s)=0$ and the condition

$$\lim_{s\to 0}\frac{[\omega(s)]^2\rho}{G}=(k_p^{B_1})^2.$$

The same limit must be performed for the coefficients $e_{ip(2)0\alpha}^{B_1A}(x,y)$.

For the solutions $\chi_{ip(j)}^{B_1A}$; $B_1=A,S$, which are antisymmetrised in the y variable, the factor $\cosh(zL_2/2)$ must be replaced by $\sinh(zL_2/2)$ in all the formulas like Eq. (18). This leads to the factors depending on k_p^A instead of k_p in the boundary conditions (13). The inversion of hyperbolic sinus and cosinus with respect to the same y symmetry leads to solutions with boundary conditions for $\chi_3 = u_y$ and $\chi_4 = \tau$ instead than u_x and σ_y.

In Eq. (29) the Fourier transform from the angular frequency ω to the time t can be performed analytically. Taking into account Eq. (10), the general solution of Eqs. (1) and (2) with the boundary conditions (3) and (4) is

$$\chi_i(x,y,t) = -\sum_{p\in\mathbf{N}} \sum_{B_1B_2} \sum_{k=-\infty}^{+\infty} \sum_{\alpha} \sum_{j=1}^{2} \int_{-\infty}^{t} c_{p(j)}^{B_1B_2}(t_1) \frac{e_{ip(j)k\alpha}^{B_1B_2}(x,y)}{\rho\omega_{k\alpha}^{B_1B_2}} \sin(\omega_{k\alpha}^{B_1B_2}(t-t_1))\,\mathrm{d}t_1. \quad (30)$$

Eq. (30) represents the more compact formulation of the solutions, but sometimes its truncations are not good approximations because of both the non-uniformity of the convergence near the edges, both the fact that it is a double series with high density of autofrequency levels. A better convergence can be obtained by substituting with expansion (26) the following equation:

$$g_k(\omega^2\rho) = g_k(0) + \frac{c_{kA}\omega^2\rho}{\omega_{kA}^2\rho(\omega^2\rho - \omega_{kA}^2\rho)} + \frac{c_{kB}\omega^2\rho}{\omega_{kB}^2\rho(\omega^2\rho - \omega_{kB}^2\rho)} + \sum_{\alpha\in\mathbf{N}} \frac{c_{k\alpha}\omega^2\rho}{\omega_{k\alpha}^2\rho(\omega^2\rho - \omega_{k\alpha}^2\rho)}. \quad (31)$$

The results are

$$\chi_{1p(1)}^{SS} = \Gamma_p(x,y) + \mathrm{H}_p(x,y) + \sum_{k=-\infty}^{+\infty} \Pi_{kp}(x,y) + \sum_{k=-\infty}^{k=+\infty} \sum_{\alpha\in\mathbf{N}} e_{1p(1)k\alpha}^{SS}(x,y) \frac{\omega^2}{(\omega_{k\alpha}^{SS})^2(\omega^2\rho - (\omega_{k\alpha}^{SS})^2\rho)}, \quad (32)$$

where

$$\Gamma_p(x,y) = \frac{\cosh(k_p y)}{\cosh\left(k_p \frac{L_2}{2}\right)} \sin(k_p x) + \Lambda \frac{\cosh(k_p y)}{\cosh\left(k_p \frac{L_2}{2}\right)} k_p \left(y\,\mathrm{tgh}(k_p y) - \frac{L_2}{2}\,\mathrm{tgh}\left(k_p \frac{L_2}{2}\right)\right) \sin(k_p x),$$

$$\mathrm{H}_p(x,y) = (-1)^{p+1} \Lambda 4 k_p\, \mathrm{sign}(x) \left(\frac{L}{2} - |x|\right) \frac{1}{\pi} \mathrm{arctg}\left(\tau^* \cos\left(\frac{\pi}{L_2} y\right)\right),$$

$$\begin{aligned}\Pi_{kp}(x,y) = &(-1)^{p+k} \frac{K_{pk}}{L_2} \frac{4\Lambda s_k^S S_k x \cosh(s_k^S x) - (4S_k + 2\Lambda L C_k s_k^S)\sinh(s_k^S x)}{\left(S_k C_k + s_k^S \frac{L}{2}\right)} \cos(s_k^S y) \\ &+ (-1)^{p+k} \frac{4}{L_2} \Lambda k_p \gamma_k(x)\, \mathrm{sign}(x) \frac{1}{s_k^S} \left(\frac{L}{2} - |x|\right) \cos(s_k^S y),\end{aligned}$$

with

$$K_{pk} = \frac{k_p s_k^2}{(k_p^2 + (s_k^S)^2)^2}, \quad S_k = \sinh\left(s_k^S \frac{L}{2}\right), \quad C_k = \cosh\left(s_k^S \frac{L}{2}\right),$$

and so on for the other $\chi_{ip(j)}^{B_1B_2}$. Here the summation of the leading terms in the series were performed analytically, as is done in Eq. (25). In Eq. (32) the static solution $\omega = 0$, may be obtained directly by neglecting the last series, therefore the resonance frequencies $\omega_{k\alpha}^{SS}$ are not involved and it is not required to find the numerical roots of Eq. (27). If the Fourier transform back to the t variable is performed taking into account Eq. (32) instead then Eq. (29), then in the right term of Eq. (30) the term

$$\sum_{B_1B_2}\sum_{p\in\mathbf{N}}\sum_{j=1}^{2}\left[\Gamma^{B_1B_2}_{p(j)}(x,y)+\mathrm{H}^{B_1B_2}_{p(j)}(x,y)+\sum_{k=-\infty}^{+\infty}\left(\Pi^{B_1B_2}_{kp(j)}(x,y)+\sum_{\alpha\in\mathbf{N}}\frac{e^{B_1B_2}_{ip(j)k\alpha}(x,y)}{\rho(\omega^{B_1B_2}_{k\alpha})^2}\right)\right]c^{B_1B_2}_{p(j)}(t), \tag{33}$$

must be summed, in which $\Gamma^{B_1B_2}_{p(j)}(x,y)$, $\mathrm{H}^{B_1B_2}_{p(j)}(x,y)$ and $\Pi^{B_1B_2}_{kp(j)}(x,y)$ are the terms corresponding to $\Gamma_p(x,y)$, $\mathrm{H}_p(x,y)$, and $\Pi_{kp}(x,y)$ related to the other symmetries B_1, B_2 and the other solutions, $(j)=1,2$. Term (33) is vanishing if the series summations are completed, but this makes the convergence faster when the series are truncated.

A faster convergence can be obtained by substituting with Eq. (31) the next pole expansion [8]:

$$g_k(\omega^2\rho)=g_k(0)+\omega^2\rho\frac{\partial g_k}{\partial(\omega^2\rho)}(0)+\frac{c_{kA}(\omega^2\rho)^2}{(\omega^2_{kA}\rho)^2(\omega^2\rho-\omega^2_{kA}\rho)}+\frac{c_{kB}(\omega^2\rho)^2}{(\omega^2_{kB}\rho)^2(\omega^2\rho-\omega^2_{kB}\rho)}$$
$$+\sum_{\alpha\in\mathbf{N}}\frac{c_{k\alpha}(\omega^2\rho)^2}{(\omega^2_{k\alpha}\rho)^2(\omega^2\rho-\omega^2_{k\alpha}\rho)}$$

which gives rather cumbersome solutions which may be useful for low frequencies, taking the corrections to the first order in $\omega^2\rho$ and neglecting the terms of the $\omega^{B_1B_2}_{k\alpha}$ resonance frequencies.

3. Results

As an example for the discussion of numerical results, the solution $\chi^{SS}_{i1(1)}(x,y,\omega)$, $i=1,\dots,5$ is considered. The related boundary conditions (13) are

$$\chi^{SS}_{11(1)}\left(x,\frac{L_2}{2},\omega\right)=\sin\left(\frac{2\pi}{L}x\right),\quad \chi^{SS}_{21(1)}\left(x,\frac{L_2}{2},\omega\right)=0,\quad |x|<\frac{L}{2}, \tag{34}$$

together with

$$\chi^{SS}_{41(1)}\left(\pm\frac{L}{2},y,\omega\right)=0,\quad \chi^{SS}_{51(1)}\left(\pm\frac{L}{2},y,\omega\right)=0,\quad |y|<\frac{L_2}{2}. \tag{35}$$

The displacement tensor components $u_x=\chi^{SS}_{11(1)}$ and $u_y=\chi^{SS}_{31(1)}$ are continuous functions, unlike the strain tensor components $\sigma_y=\chi^{SS}_{21(1)}$, $\tau=\chi^{SS}_{41(1)}$ and $\sigma_x=\chi^{SS}_{51(1)}$ which are discontinuous in the vertices. The discontinuities can be investigated by means of analytical methods. Eq. (14), together with boundary conditions (34), implies that

$$\chi^{SS}_{51(1)}\left(x,\frac{L_2}{2},\omega\right)=\frac{4G(G+\lambda)}{(\lambda+2G)}\frac{2\pi}{L}\cos\left(\frac{2\pi}{L}x\right),$$

and therefore that

$$\lim_{x\to\frac{L}{2}}\chi^{SS}_{51(1)}\left(x,\frac{L_2}{2},\omega\right)=\frac{-4G(G+\lambda)}{\lambda+2G}\frac{2\pi}{L}.$$

At the same time, it follows from Eq. (35) that

$$\lim_{y\to\frac{L_2}{2}} \chi^{SS}_{51(1)}\left(\frac{L}{2}, y, \omega\right) = 0.$$

Therefore, the vertex value $\chi^{SS}_{51(1)}(L/2, L_2/2, \omega)$ is not determined. Likewise, the following discontinuities can be found taking into account that in Eqs. (25) the summations are uniformly convergent:

$$\lim_{x\to\frac{L}{2}} \chi^{SS}_{21(1)}\left(x, \frac{L_2}{2}, \omega\right) = 0, \quad \lim_{y\to\frac{L_2}{2}} \chi^{SS}_{21(1)}\left(\frac{L}{2}, y, \omega\right) = \frac{-4G(\lambda+G)}{(\lambda+2G)}k_1,$$

$$\lim_{x\to\frac{L}{2}} \chi^{SS}_{41(1)}\left(x, \frac{L_2}{2}, \omega\right) = \frac{-8G(\lambda+G)}{(\lambda+2G)}\frac{k_1}{\pi}, \quad \lim_{y\to\frac{L_2}{2}} \chi^{SS}_{41(1)}\left(\frac{L}{2}, y, \omega\right) = 0.$$

In spite of these discontinuities, the solution has a well defined physical meaning. It must be referred to a rectangle with the vertices removed as in Fig. 1. Let ε be the length of the cut $l_\varepsilon = \overline{\mathbf{AB}}$, h the thickness of the rectangle in the z direction, n_x and n_y the components of the vector normal to the edge. Then the components f_x, f_y of the external force acting on the cut are given by

$$f_x = h\int_{l_\varepsilon} \sigma_x(x, y, \omega)n_x + \tau(x, y, \omega)n_y \,\mathrm{d}l,$$

$$f_y = h\int_{l_\varepsilon} \sigma_y(x, y, \omega)n_y + \tau(x, y, \omega)n_x \,\mathrm{d}l,$$

and are vanishing as $\varepsilon \to 0$ due to the finiteness of the discontinuities in the stress tensor. As a consequence, the boundary conditions (34) and (35) are preserved in the limit. However the convergence of the summations in the neighbourhood of the discontinuities is slow, and therefore numerical results obtained by series truncation must not be taken too close to the removed vertices.

In the following, results are given in terms of non-dimensional quantities. Co-ordinates are represented by $\hat{x} = x/L$, $\hat{y} = y/L_2$. Angular frequency is represented by the quantity $\hat{\omega} = \sqrt{\rho/Y}L\omega$, where Y is the Young modulus, given by

$$Y = G\frac{3\lambda^* + 2G}{\lambda^* + 2G} = 4G\frac{\lambda + G}{\lambda + 2G}.$$

Fig. 1. Removal of the vertices from the rectangle.

Table 1
Non-dimensional values for the lowest resonance frequencies of the solution symmetric in the x and y direction

	$\hat{\omega}^{SS}_{k\alpha}$								
$k\backslash\alpha$	1	2	3	4	5	6	7	8	9
0	3.396	4.814	6.275	8.428	10.715	12.528	16.032	17.066	20.234
1	8.138	10.628	13.382	14.442	15.555	16.906	18.807	20.859	22.668
2	13.530	15.769	17.812	20.441	23.223	24.071	24.875	26.302	27.603
3	18.941	21.403	22.882	24.991	27.525	30.317	33.056	33.699	34.341
4	24.353	27.200	28.329	30.029	32.169	34.642	37.357	40.214	42.861
5	29.765	33.065	33.966	35.368	37.187	39.346	41.779	44.427	47.234

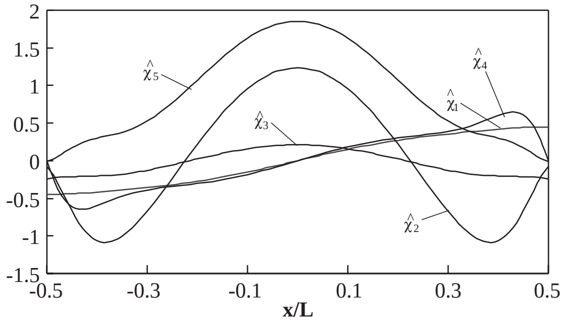

Fig. 2. The shape of the components of the fundamental solution with symmetry SS; $\hat{\chi}^{SS}_{i1(1)}$; $i = 1, \ldots, 5$, as a function of the transversal coordinate x, for $\hat{\omega} = 1.325$ and $\hat{y} = 0.29$.

To deal with non-dimensional stress and displacement tensors, the first of boundary conditions (34) must be rewritten as $u_x = \chi^{SS}_{11(1)} = A_0 \sin\left(\frac{2\pi}{L} x\right)$, in which the amplitude A_0 satisfies $A_0 = 1$ and has the dimension of a length. Then the non-dimensional quantities $\hat{\chi}_1 = \chi^{SS}_{11(1)}/A_0$ $\hat{\chi}_2 = (L/YA_0)\chi^{SS}_{21(1)}$ $\hat{\chi}_3 = \chi^{SS}_{31(1)}/A_0$ $\hat{\chi}_4 = (L/YA_0)\chi^{SS}_{41(1)}$ and $\hat{\chi}_5 = (L/YA_0)\chi^{SS}_{51(1)}$ are defined. The shape of the rectangle and the elastic properties of the material are determined by the non-dimensional values $L_2/L = 0.6725$ and $\lambda/G = 0.6$. The lower resonance frequencies are shown in Table 1. The graph in Fig. 2 represents the solution for $\hat{\omega} = 1.325$ and $\hat{y} = 0.29$, taken from Eq. (25) with the k summation truncated to the value $k_{max} = 65$. To check how the numerical approximation fulfils the equations of the elasticity, it is useful to introduce the following non-dimensional quantities

$$\hat{g}_1 = |\partial_y \chi^{SS}_{11(1)} + \partial_x \chi^{SS}_{31(1)} - \chi^{SS}_{41(1)}/G| L/A_0,$$

$$\hat{g}_2 = |\partial_y \chi^{SS}_{21(1)} + \rho\omega^2 \chi^{SS}_{31(1)} + \partial_x \chi^{SS}_{41(1)}/G| \frac{L^2}{A_0 Y},$$

$$\hat{g}_3 = \left| \partial_y \chi^{SS}_{31(1)} + \frac{\lambda}{\lambda + 2G} \partial_x \chi^{SS}_{11(1)} - \frac{1}{\lambda + 2G} \chi^{SS}_{21(1)} \right| L/A_0,$$

$$\hat{g}_4 = \left| \partial_y \chi^{SS}_{41(1)} + \left(\frac{4G(\lambda + G)}{\lambda + 2G} \partial_x^2 + \rho\omega^2 \right) \chi^{SS}_{11(1)} + \frac{\lambda}{\lambda + 2G} \partial_x \chi^{SS}_{21(1)} \right| \frac{L^2}{A_0 Y}.$$

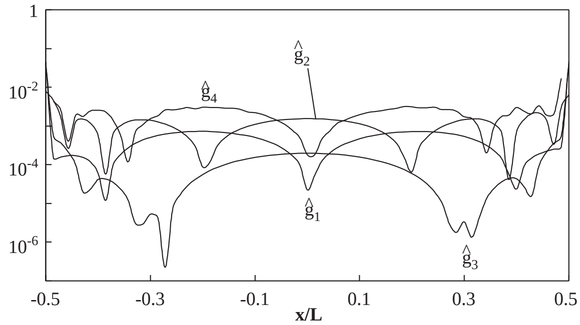

Fig. 3. The graph shows how the solution $\hat{\chi}^{SS}_{i1(1)}$; $i = 1, \ldots, 5$, truncated in its series expansion, fulfils the equations of elasticity $\hat{g}_i = 0$, $i = 1, \ldots, 4$ for $\hat{\omega} = 1.325$, $\hat{y} = 0.29$, $-0.5 \leqslant \hat{x} \leqslant 0.5$.

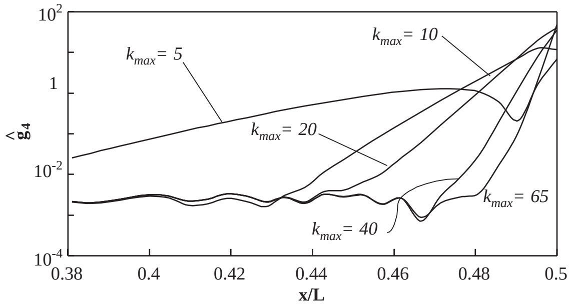

Fig. 4. Fulfilment of the $\hat{g}_4 = 0$ equation (see Fig. 3) in proximity to the edge of the rectangle $x/L = 0.5$. The series in the solution have been truncated to $k_{max}. = 5, 10, 20, 40$ and 65.

Then Eqs. (15) can be written as $\hat{g}_i = 0$, $i = 1, 2, 3, 4$. These equations, together with Eq. (14), are equivalent to the elasticity equations (1) and (2). Hereafter derivatives ∂_x and ∂_y are calculated by means of finite increments on the x and y variables. As was mentioned in Section 2, if $|x| < L/2$, the convergence in the k series is determined by the goniometrical factors which behave as

$$e^{-\frac{2\pi}{L_2} k \left(\frac{L}{2} - |x|\right)},$$

if k tends to infinity; meanwhile, if $x = \pm L/2$, the convergence is determined by algebraic factors and can be expressed in powers of k. It follows that the quantity $\hat{g}_4$ is convergent in the internal points, but not on the edge, due to the presence of the term ∂_x^2 which behaves like $k^2(2\pi/L_2)^2$. For the other $\hat{g}_i$ it must be taken into account that close to the edge, the ∂_x term makes the convergence weaker than that of the $\hat{\chi}^{SS}_{i1(1)}$. Therefore, the errors in the $\hat{g}$ variables are greater than those in the $\hat{\chi}$ (as can be confirmed by comparing the results in Fig. 3 with the stability in Fig. 5). Numerical results for the $\hat{g}_i$ if $\hat{y} = 0.29$ obtained by truncation of the series to $k_{max} = 65$ are shown in Fig. 3. The $\hat{g}_4$ values corresponding to $x = \pm L/2$ are not reported in Fig. 3 because they are not convergent. The convergence of $\hat{g}_4$ to zero near the edge and its k_{max} dependence is shown in Fig. 4. The solutions taken from Eq. (25) by truncation have a convergence which is faster inside than on the edge. The graphics in Fig. 5 show the convergence in k_{max} of $\hat{\chi}^{SS}_{11(1)}$, $\hat{\chi}^{SS}_{21(1)}$ and $\hat{\chi}^{SS}_{31(1)}$ on

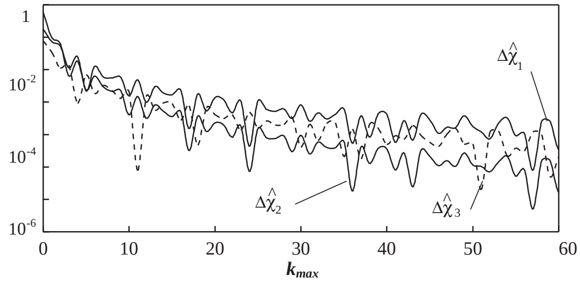

Fig. 5. Convergence of the components of the solution $\hat{\chi}^{SS}_{i1(1)}$ on the edge $x = L/2$ as a function of the truncation value k_{max}, $\hat{y} = 0.29$, $\hat{\omega} = 1.325$. Absolute value of the errors are reported: $\Delta\hat{\chi}_i = |\hat{\chi}_i(k_{max}) - \hat{\chi}_i(k_{max} = 65)|$.

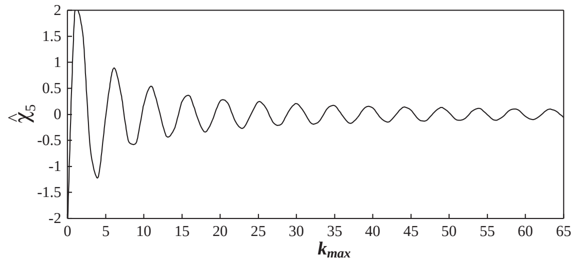

Fig. 6. Convergence of $\hat{\chi}^{SS}_{51(1)} \equiv \hat{\sigma}_x$ to the boundary condition $\hat{\sigma}_x = 0$ on the edge $x = L/2$ as a function of the truncation value k_{max}.

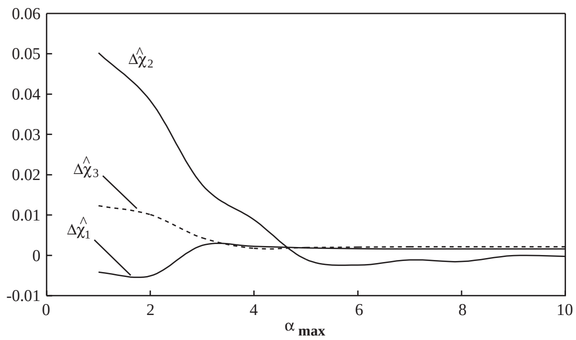

Fig. 7. Error in the approximation of the fundamental solution on the edge $x = L/2$ as a function of the truncation value α_{max}; $k_{max} = 10$, $\hat{y} = 0.29$, $\hat{\omega} = 1.325$, according to Eq. (32). Errors $\Delta\hat{\chi}_i$ are computed from $\Delta\hat{\chi}_i = \hat{\chi}_i(\alpha_{max}) - \hat{\chi}_i(k_{max} = 65)$ in which $\hat{\chi}_i(k_{max} = 65)$ is taken from Eq. (25) and is considered equivalent to the unknown exact value.

the edge $x = L/2$ for $\hat{y} = 0.29$. Errors are evaluated subtracting the $k_{max} = 65$ approximation instead of the unknown exact value. Errors in $\hat{\chi}^{SS}_{41(1)}$ are not reported in Fig. 5 since this component vanishes exactly for every order of approximation. Due to the ∂_x term in Eq. (14), the $\hat{\chi}^{SS}_{51(1)}$ convergence on the edge is slow, as shown in Fig. 6. This is not a problem for the applications since the edge value is known to be vanishing from the boundary conditions, while for the values in the inner points the convergence is faster. Fig. 7 shows the approximation for truncation in

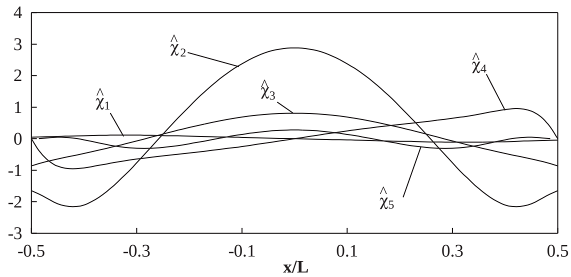

Fig. 8. The shape of the components of the fundamental solution with symmetry SS; $\hat{\chi}_{i1(1)}^{SS}$; $i = 1, \ldots, 5$, as a function of the transversal coordinate x, for $\hat{\omega} = 3.773$, $\hat{y} = 0.29$.

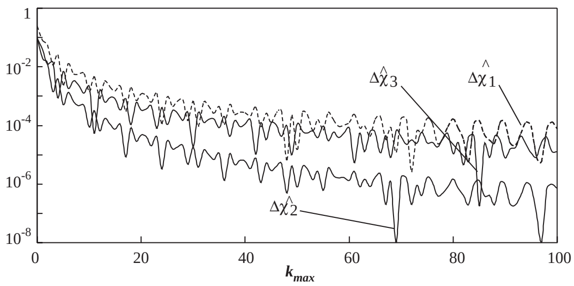

Fig. 9. Convergence of the components of the solution $\hat{\chi}_{i1(1)}^{SS}$ on the edge $x = L/2$ as a function of the truncation value k_{max}, $\hat{y} = 0.29$; $\hat{\omega} = 3.77$. Absolute values where used: $\Delta\hat{\chi}_i = |\hat{\chi}_i(k_{max}) - \hat{\chi}_i(k_{max} = 101)|$.

Eq. (32), for $k_{max} = 10$ and $0 < \alpha_{max} < 10$, α_{max} being the value to which the α series is truncated. If $\alpha_{max} > 8$ the approximation is in the same order as that for $k_{max} = 10$ in Fig. 5. Therefore increasing α_{max} values will not improve convergence if there are not increments in k_{max}. In Figs. 8 and 9 the value $\hat{\omega} = 3.773$ was investigated while the other parameters are the same as in Fig. 2. This is to check the approximation for frequencies exceeding the first resonance of Table 1. Fig. 8 shows the solution for $k_{max} = 75$ while Fig. 9 shows the convergence on the edge for $0 < k_{max} < 100$.

4. Concluding remarks

An analytical solution involving double series summation, both on the resonance frequencies and on the length wave numbers of the free edges has been derived. Some formulations of the solution which are more suitable for the static problem, for low frequencies and for numerical series truncation are discussed. A 10-term truncation is sufficient to give an approximation in the order of one part per thousand. If the Fourier transform must be performed back to the time variable, then a 50-term truncation is required to achieve the same result. The relevance of the solution presented in this paper for the applications is briefly discussed. Suppose that the elastic rectangle acts as a damper lying between two extended bodies A_1 and A_2 which are in contact with the $y = L_2/2$ and $y = -L_2/2$ edges. This is the situation for which the free boundary conditions on the other two edges were introduced. For the determination of the time evolution of the system, the rectangle cannot be replaced by a simple spring as long as the frequency of the motion of the system is within the range of the resonance frequencies of the rectangle or if the asymmetry with respect to the $x = 0$ axis of the rectangle makes the exchange of torque with the A_1 and A_2 bodies to be not negligible. This situation occurs, for instance, in the case in which the rectangle models the felt lying between the wooden core of a piano hammer and the string, as long as the longitudinal string vibrations are involved. To this extent, the elastic rectangle can be presented as a direct generalization of the concept of spring. If the inputs of the rectangle from A_1 and A_2 are known at the time t, that is if the boundary conditions $P_y(t) = \sigma_y(t)$ and $s_x(t) = u_x(t)$ are given, then Eq. (30) gives the outputs of the rectangle $u_x(t + \Delta t)$ and $\tau(t + \Delta t)$ at the time $t + \Delta t$. Taking these outputs, the next stage inputs $P_y(t + \Delta t) = \sigma_y(t + \Delta t)$ and $s_x(t + \Delta t) = u_x(t + \Delta t)$ can be determined from the dynamical properties of the bodies A_1 and A_2. Then the dynamics of the whole system is obtained using a time step integration. Therefore there is no need to introduce a space lattice model for the rectangle and there are no problems in convergence if the frequencies are close to the resonance frequencies of the rectangle, since Eq. (30) behaves well even for those frequencies. If Eq. (30) is re-written as

$$\chi(t) = -\sum_{(R)} \int_{-\infty}^{t} c(t_1) \frac{e_R}{\rho\omega_R} \sin(\omega_R(t - t_1))\, \mathrm{d}t_1,$$

where $c(t_1)$ represents the inputs from A_1 and A2, $\chi(t)$ represents the outputs of the rectangle, e_R and ω_R represent the amplitudes and resonance frequencies $e^{B_1B_2}_{ip(j)k\alpha}$ and $\omega^{B_1B_2}_{k\alpha}$, which can be computed once and for all before the time iteration to be performed, then it is apparent that the time iteration can be performed by the simple recursion formulae

$$\chi(t + \Delta t) = \chi(t) - \sum_{(R)} \frac{e_R}{\rho} (\cos(\omega_R t) I_{1R}(t) + \sin(\omega_R t) I_{2R}(t))\Delta t,$$

$$I_{1R}(t + \Delta t) = I_{1R}(t) + \cos(\omega_R t) c(t)\Delta t,$$

$$I_{2R}(t + \Delta t) = I_{2R}(t) + \sin(\omega_R t) c(t)\Delta t,$$

which require only a short computation time when employed in a numerical simulation. For a better convergence a term (33) can be added. In this term the coefficient of $c(t)$ does not depend on t, then its computation time will also be fast.

Appendix A

This Appendix provides the proof of the modified Fourier formula:

$$f(x) = \sum_{k=-\infty}^{k=+\infty} a_k \mathrm{e}^{\mathrm{i}\left(k+\frac{1}{2}\right)x} \quad \text{if } |x| < \pi,$$

where

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}\left(k+\frac{1}{2}\right)x'} f(x')\, \mathrm{d}x'.$$

This formula is required for the Fourier expansions in Eqs. (10). Taking into account the identity

$$\sum_{-n}^{n} \mathrm{e}^{\mathrm{i}\left(k+\frac{1}{2}\right)x} = \mathrm{e}^{\mathrm{i}\frac{x}{2}} \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)},$$

it is apparent that

$$\sum_{-n}^{n} a_k \mathrm{e}^{\mathrm{i}\left(k+\frac{1}{2}\right)x} - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\frac{\mathrm{i}}{2}(x-x')} \frac{\sin\left(\left(n+\frac{1}{2}\right)(x-x')\right)}{\sin\left(\frac{x-x'}{2}\right)} f(x')\,\mathrm{d}x' - f(x). \tag{A.1}$$

A straightforward calculation gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\frac{\mathrm{i}}{2}(x-x')} \frac{\sin\left(\left(n+\frac{1}{2}\right)(x-x')\right)}{\sin\left(\frac{x-x'}{2}\right)}\,\mathrm{d}x' = \sum_{-n}^{n} \frac{1}{2\pi} \mathrm{e}^{\mathrm{i}\left(k+\frac{1}{2}\right)x} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}\left(k+\frac{1}{2}\right)x'}\,\mathrm{d}x' = \sum_{-n}^{n} \frac{1}{\pi} \frac{(-1)^k}{\left(k+\frac{1}{2}\right)} \mathrm{e}^{\mathrm{i}\left(k+\frac{1}{2}\right)x}.$$

The latter series may be evaluated by considering

$$\varphi(z) = \frac{1}{z+\frac{1}{2}} \frac{\mathrm{e}^{\mathrm{i}\left(z+\frac{1}{2}\right)x}}{\sin(\pi z)},$$

and then using the residue theory methods for the summation of all the residues, which gives

$$\lim_{n\to\infty} \sum_{-n}^{n} \frac{1}{\pi} \frac{(-1)^k}{\left(k+\frac{1}{2}\right)} \mathrm{e}^{\mathrm{i}\left(k+\frac{1}{2}\right)x} = 1.$$

Thus Eq. (A.1) reads

$$\lim_{n\to\infty} \sum_{-n}^{n} a_k \mathrm{e}^{\mathrm{i}\left(k+\frac{1}{2}\right)x} - f(x) = \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\frac{\mathrm{i}}{2}(x-x')} \frac{\sin\left(\left(n+\frac{1}{2}\right)(x-x')\right)}{\sin\left(\frac{x-x'}{2}\right)} (f(x')-f(x))\,\mathrm{d}x'.$$

This expression may be rearranged as

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} \sin\left(\left(n+\frac{1}{2}\right)y\right)\psi_x(y)\,\mathrm{d}y, \tag{A.2}$$

where $\psi_x(y) = \frac{\mathrm{e}^{\frac{\mathrm{i}y}{2}}}{\sin(y/2)}(f(x-y)-f(x))$ is a limited function in the neighborhood of $y=0$. It is well known (see for example, Ref. [9]) that the limit in (A.2) is vanishing if the function $\psi_x(y)$ admits a finite number of intervals which is monotone and has a finite number of discontinuities.

Appendix B

In order to prove that all the roots of Eqs. (27) are real, it must be written in terms of the variable $\zeta = \omega^2\rho$:

$$\phi^S(\zeta) = \phi_1^S(\zeta) + \phi_2^S(\zeta) = 0,$$

where

$$\phi_1^S(\zeta) = \xi_1 \operatorname{tg}\left(\xi_1 \frac{L}{2}\right), \quad \phi_2^S(\zeta) = \frac{\left(\frac{\zeta}{2G} - (s_k^S)^2\right)^2}{s_k^2 \xi_2} \operatorname{tg}\left(\xi_2 \frac{L}{2}\right),$$

$$\xi_1(\zeta) = \sqrt{\frac{\zeta}{\lambda + 2G} - (s_k^S)^2}, \quad \xi_2(\zeta) = \sqrt{\frac{\zeta}{G} - (s_k^S)^2}.$$

As a consequence of the residue theory

$$\int \frac{\phi^{S'}(\zeta)}{\phi^S(\zeta)} \mathrm{d}\zeta = n_R(\phi^S) - n_P(\phi^S), \tag{B.1}$$

where the integral is taken on a closed path, n_R and n_P are the number of roots and poles of the function ϕ^S delimited by the path. If the integration path is rectangular, with edges $\zeta = \pm\zeta_0$ and $\zeta = \pm \mathrm{i}\zeta_0$, with $\zeta_0 \in \mathbf{R}$ such that $\zeta_0 \gg (\lambda + 2G)z_k^2$, the inequality can be stated as

$$|\phi_1^S(\zeta)| \ll |\phi_2^S(\zeta)|, \tag{B.2}$$

if it is assumed not to be $\xi_1(\zeta_0)L/2\pi \cong n + \frac{1}{2}$ or $\xi_2(\zeta_0)L/2\pi \cong n$ with $n \in \mathbf{N}$. So the inequality

$$\left| \frac{\phi_2^S(\zeta)}{\phi^S(\zeta)} - 1 \right| < \frac{1}{2}. \tag{B.3}$$

holds on the integration path. Now consider the equation

$$\int \frac{\phi^{S'}(\zeta)}{\phi^S(\zeta)} \mathrm{d}\zeta = \int \mathrm{d}\log(\phi^S(\zeta)) = \Delta \log(\phi^S),$$

where Δ is the variation of the function (which is not a single-valued one) on the closed path. It is apparent that $\Delta\log(\phi_2^S) = \Delta\log(\phi^S) + \Delta\log(\phi_2^S/\phi^S) = \Delta\log(\phi^S)$, as a consequence of $\Delta\log(\phi_2^S/\phi^S) = \mathrm{i}\Delta\arg(\phi_2^S/\phi^S) = 0$, where the variation of the argument is vanishing, since the quantity $\phi_2^S(\zeta)/\phi^S(\zeta)$ does not go round the origin, being confined in the neighborhood of the $z = 1$ of the complex plane as can be argued from the condition (B.3). It follows

$$\int \frac{\phi^{S'}(\zeta)}{\phi^S(\zeta)} \mathrm{d}\zeta = \int \frac{\phi_2^{S'}(\zeta)}{\phi_2^S(\zeta)} \mathrm{d}\zeta,$$

and

$$n_R(\phi^S) - n_P(\phi^S) = n_R(\phi_2^S) - n_P(\phi_2^S). \tag{B.4}$$

It can be easily verified that $n_R(\phi_2^S) = 2 + \left[\frac{\xi_2(\zeta_0)L}{2\pi}\right]$ where the 2 is determined by the double root of the factor $(\zeta/2G - s_k^2)^2$, and $[x]$ represents the greatest integer which is less then x. It is apparent that

$$n_P(\phi_2^S) = \left[\frac{\xi_2(\zeta_0)L}{2\pi} + \frac{1}{2}\right] \quad \text{and} \quad n_P(\phi^S) = \left[\frac{\xi_1(\zeta_0)L}{2\pi} + \frac{1}{2}\right] + \left[\frac{\xi_2(\zeta_0)L}{2\pi} + \frac{1}{2}\right],$$

therefore, taking into account equation (B.4) the number of roots of ϕ^S is

$$n_R(\phi^S) = 2 + \left[\frac{\xi_1(\zeta_0)L}{2\pi} + \frac{1}{2}\right] + \left[\frac{\xi_2(\zeta_0)L}{2\pi}\right]. \tag{B.5}$$

Now the number of the real roots of the same function is to be found. If

$$\xi_1(\zeta) \to \frac{2\pi}{L}\left(n + \frac{1}{2}\right)^- \quad \text{or} \quad \xi_2(\zeta) \to \frac{2\pi}{L}\left(n + \frac{1}{2}\right)^-, \quad \text{with } n \in \mathbf{N},$$

then $\phi^S(\zeta) \to +\infty$, while if

$$\xi_1(\zeta) \to \frac{2\pi}{L}\left(n + \frac{1}{2}\right)^+ \quad \text{or} \quad \xi_2(\zeta) \to \frac{2\pi}{L}\left(n + \frac{1}{2}\right)^+, \quad \text{with } n \in \mathbf{N},$$

then $\phi^S(\zeta) \to -\infty$.

This means that in the intervals $]a_i, b_i[$ in which the function ϕ^s is continuous, the left-hand term always has the value $-\infty$: $\phi(a_i) = -\infty$ and the right-hand term always has the value $+\infty$: $\phi(b_i) = +\infty$; therefore there is at least one root for every $+\infty$. As far as the first $+\infty$ is concerned, which occurs for $\xi_2(\zeta) = \pi/L$, it should be noted that in the former value of the variable $\zeta = Gs_k^2$, the function takes a negative value $\phi^S(\zeta) < 0$, so that even in this case there is a former root. It follows, for the number n_{RR} of real roots:

$$n_{RR}(\phi^S) \geqslant \left[\frac{\xi_1(\zeta_0)L}{2\pi} + \frac{1}{2}\right] + \left[\frac{\xi_2(\zeta_0)L}{2\pi} + \frac{1}{2}\right].$$

As a consequence of Eq. (B.2), it is known that the root that precedes the last infinite is near to the point $\xi_2(\zeta)L/2\pi = n$, so it is possible to substitute

$$[\xi_2(\zeta_0)L/2\pi + 1/2] \quad \text{by} \ [\xi_2(\zeta_0)L/2\pi] + 1.$$

It is apparent that another root may be found for $\zeta = 0$, therefore

$$n_{RR}(\phi^S) \geqslant 2 + \left[\frac{\xi_1(\zeta_0)L}{2\pi} + \frac{1}{2}\right] + \left[\frac{\xi_2(\zeta_0)L}{2\pi}\right].$$

Comparing this with equation (B.5) one can infer that all the roots are real.

References

[1] A.W. Leissa, The free vibration of rectangular plates, *Journal of Sound and Vibration* 31 (1973) 257–293.

[2] D.J. Gorman, *Free Vibrations of Rectangular Plates*, North-Holland, Amsterdam, 1982.

[3] A.W. Leissa, *Vibrations of Plates*, Acoustical Society of America Publications, New York, 1993.

[4] A.W. Leissa, A higher order shear deformation theory for the vibration of thick plates, *Journal of Sound and Vibration* 170 (1992) 545–555.

[5] N.I. Muskhelishvili, *Some Basic Problems in the Theory of Elasticity*, Nauka, Moskow, 1966 (in Russian; These results are presented in English in: V.Z. Parton, P.I. Perlin, *Mathematical Methods of the Theory of Elasticity*, Vol. 2, Mir Publishers, Moscow, 1984, pp. 63–80.).

[6] J.R.M. Radok, On the solution of problems of dynamic plane elasticity, *The Quarterly Journal of Mechanics and Applied Mathematics* 14 (1956) 289–298.

[7] V.Z. Parton, P.I. Perlin, *Mathematical Methods of the Theory of Elasticity*, Vol. 1, Mir Publishers, Moscow, 1984, pp. 253–255; or L.D. Landau, *Course of Theoretical Physics*, Vol. 7, *Theory of Elasticity*, Pergamon Press, London, 1959, pp. 52–54.

[8] V.I. Smirnov, *A Course in Higher Mathematics*, Vol. 3, part two: Complex Variables/Special Functions, Pergamon Press, London, 1964, pp. 248–251.

[9] V.I. Smirnov, *A Course in Higher Mathematics*, Vol. 2: Advanced Calculus, Pergamon Press, London, 1964, pp. 434–438.